

Bethe Ansatz solution of the open XX spin chain with nondiagonal boundary terms

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Abstract

We consider the integrable open XX quantum spin chain with nondiagonal boundary terms. We derive an exact inversion identity, using which we obtain the eigenvalues of the transfer matrix and the Bethe Ansatz equations. For generic values of the boundary parameters, the Bethe Ansatz solution is formulated in terms of Jacobian elliptic functions.

1 Introduction

We consider in this article the open XX quantum spin chain with nondiagonal boundary terms, defined by the Hamiltonian

$$\begin{aligned} \mathcal{H} = & \frac{1}{2} \left\{ \sum_{n=1}^{N-1} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y) \right. \\ & \left. + i \coth \xi_- \sigma_1^z + \frac{2i\kappa_-}{\sinh \xi_-} \sigma_1^x - i \coth \xi_+ \sigma_N^z + \frac{2i\kappa_+}{\sinh \xi_+} \sigma_N^x \right\}, \end{aligned} \quad (1.1)$$

where $\sigma^x, \sigma^y, \sigma^z$ are the standard Pauli matrices, and ξ_{\pm}, κ_{\pm} are arbitrary boundary parameters. This model is known to be integrable [1, 2, 3]. It has been investigated [4, 5] using a fermionization technique [6, 7], suitably adapted to accommodate boundary terms [8, 9]. However, this model has until now resisted a direct Bethe Ansatz solution due to the absence of a simple reference (pseudovacuum) state.¹ Such a solution is desirable for a number of reasons. First, the open XX chain is a special case of integrable open XXZ and open XYZ chains, which should also admit Bethe Ansatz solutions but which cannot be solved by fermionization. Second, Bethe Ansatz solutions are particularly well-suited for investigating physical properties, such as ground state, low-lying excitations, scattering matrices, etc. In particular, the Bethe Ansatz approach avoids the projection mechanism [5] which can be implemented only in special cases. (See [4, 5] and references therein for discussions of interesting physical applications of the open XX spin chain.) Finally, the Sklyanin transfer matrix for the open XX spin chain is closely related to the Yang matrix [11, 12, 13] for a large class of integrable $\mathcal{N} = 2$ supersymmetric quantum field theories with boundary [14, 15]. Diagonalization of this matrix is a key step in formulating the thermodynamic Bethe Ansatz equations for these $\mathcal{N} = 2$ supersymmetric models.

In this paper we derive an exact inversion identity for the model (1.1), using which we obtain the eigenvalues of the transfer matrix and the Bethe Ansatz equations. (This approach does not completely circumvent the problem of not having a reference state, since the eigenvectors are not determined.) We obtain the inversion identity using the open-chain fusion formula [16] together with the remarkable fact that, for the open XX spin chain/ 6-vertex free-Fermion model, the fused transfer matrix is proportional to the identity matrix. A similar strategy has recently been used [13] to solve the open 8-vertex free-Fermion [17] model, which corresponds to the case of $\mathcal{N} = 1$ supersymmetry. These techniques are generalizations of those which have been developed for closed spin chains [18, 19, 20, 21].

Even though the transfer matrix is constructed entirely from hyperbolic functions, we find

¹For the special case of diagonal boundary terms (i.e., $\kappa_{\pm} = 0$), a simple pseudovacuum state does exist, and a Bethe Ansatz solution is known [10, 1].

that the Bethe Ansatz solution is formulated in terms of Jacobian elliptic functions for generic values of the boundary parameters. (In contrast, for the XYZ chain [18], such functions appear already in the transfer matrix.) For special values of the boundary parameters, the elliptic functions degenerate into ordinary hyperbolic or trigonometric functions.

The outline of this paper is as follows. In Section 2, we review the construction of the Sklyanin transfer matrix for the open XX chain, and derive some of its important properties. We derive in Section 3 the inversion identity, which we then use in Section 4 to determine the Bethe Ansatz solution. In Section 5, we investigate some special cases in which the solution can be expressed in terms of ordinary hyperbolic functions. In particular, we verify that our solution is similar to the known one [10, 1] for the case of diagonal boundary terms. In Section 6, we conclude with a brief discussion of some possible directions for future work.

2 Transfer matrix

The object of central importance in the construction of integrable quantum spin chains is the one-parameter family of commuting matrices called the transfer matrix. The transfer matrix for an open chain is made from two basic building blocks, called R (bulk) and K (boundary) matrices.

An R matrix is a solution of the Yang-Baxter equation

$$R_{12}(u-v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u-v). \quad (2.1)$$

(See, e.g., [21, 22, 23].) The XX spin chain is a special case of the XXZ spin chain, corresponding (in the notation of [1]) to the anisotropy value $\eta = \frac{i\pi}{2}$. The R matrix is therefore the 4×4 matrix

$$R(u) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad (2.2)$$

with matrix elements

$$a = \sinh(u + \frac{i\pi}{2}), \quad b = \sinh u, \quad c = \sinh \frac{i\pi}{2}, \quad (2.3)$$

which satisfy the free-Fermion condition

$$a^2 + b^2 = c^2. \quad (2.4)$$

This R matrix has the symmetry properties

$$R_{12}(u) = \mathcal{P}_{12} R_{12}(u) \mathcal{P}_{12} = R_{12}(u)^{t_1 t_2}, \quad (2.5)$$

where \mathcal{P}_{12} is the permutation matrix and t denotes transpose. Moreover, it satisfies the unitarity relation

$$R_{12}(u) R_{12}(-u) = \zeta(u) \mathbb{I}, \quad \zeta(u) = -\cosh^2 u, \quad (2.6)$$

and the crossing relation

$$R_{12}(u) = V_1 R_{12}(-u - \rho)^{t_2} V_1, \quad (2.7)$$

with

$$\rho = -\frac{i\pi}{2}, \quad V = \sigma^x. \quad (2.8)$$

Finally, it has the periodicity property

$$R_{12}(u + i\pi) = -\sigma_2^z R_{12}(u) \sigma_2^z = -\sigma_1^z R_{12}(u) \sigma_1^z. \quad (2.9)$$

The matrix $K^-(u)$ is a solution of the boundary Yang-Baxter equation [24]

$$R_{12}(u - v) K_1^-(u) R_{21}(u + v) K_2^-(v) = K_2^-(v) R_{12}(u + v) K_1^-(u) R_{21}(u - v). \quad (2.10)$$

We consider here the following 2×2 matrix [2, 3]

$$K^-(u) = \begin{pmatrix} \sinh(\xi_- + u) & \kappa_- \sinh 2u \\ \kappa_- \sinh 2u & \sinh(\xi_- - u) \end{pmatrix}, \quad (2.11)$$

which evidently depends on two boundary parameters ξ_- , κ_- . We set the matrix $K^+(u)$ to be $K^-(-u - \rho)$ with (ξ_-, κ_-) replaced by (ξ_+, κ_+) ; i.e.,

$$K^+(u) = \begin{pmatrix} i \cosh(\xi_+ - u) & \kappa_+ \sinh 2u \\ \kappa_+ \sinh 2u & -i \cosh(\xi_+ + u) \end{pmatrix}, \quad (2.12)$$

We shall often use an alternative [3] set of boundary parameters $(\eta_{\mp}, \vartheta_{\mp})$ which is related to the set $(\xi_{\mp}, \kappa_{\mp})$ by

$$\cos \eta_{\mp} \cosh \vartheta_{\mp} = \frac{i}{2\kappa_{\mp}} \sinh \xi_{\mp}, \quad \cos^2 \eta_{\mp} + \cosh^2 \vartheta_{\mp} = 1 + \frac{1}{4\kappa_{\mp}^2}. \quad (2.13)$$

The K matrices have the periodicity property

$$K^{\mp}(u + i\pi) = -\sigma^z K^{\mp}(u) \sigma^z. \quad (2.14)$$

The transfer matrix $t(u)$ for an open chain of N spins is given by [1]

$$t(u) = \text{tr}_0 K_0^+(u) T_0(u) K_0^-(u) \hat{T}_0(u), \quad (2.15)$$

where tr_0 denotes trace over the “auxiliary space” 0, and $T_0(u)$, $\hat{T}_0(u)$ are so-called monodromy matrices²

$$T_0(u) = R_{0N}(u) \cdots R_{01}(u), \quad \hat{T}_0(u) = R_{10}(u) \cdots R_{N0}(u). \quad (2.16)$$

Indeed, Sklyanin has shown that $t(u)$ constitutes a one-parameter commutative family of matrices

$$[t(u), t(v)] = 0. \quad (2.17)$$

Typically, the Hamiltonian is proportional to the first derivative of the transfer matrix $t'(0)$ [1]. However, this quantity is trivial for the XX model, due to the fact $\text{tr} K^+(0) = 0$. In order to obtain the Hamiltonian, we must go to the second derivative [25]. We find

$$\begin{aligned} \mathcal{H} &= \frac{t''(0)}{4(-1)^{N+1}i \sinh \xi_- \text{tr} K^{+'}(0)} \\ &= \sum_{n=1}^{N-1} H_{n,n+1} + \frac{i}{2 \sinh \xi_-} K_1^{-'}(0) + \frac{\text{tr}_0 K_0^{+'}(0) H_{N0} - i \text{tr}_0 K_0^{+}(0) H_{N0}^2}{\text{tr} K^{+'}(0)}, \end{aligned} \quad (2.18)$$

where $H_{n,n+1} = \mathcal{P}_{n,n+1} R'_{n,n+1}(0)$, and we have made use of the facts $K^-(0) = \sinh \xi_- \mathbb{I}$ and $\text{tr}_0 K_0^+(0) H_{N0} = \text{tr} K^{+''}(0) = 0$. By explicitly evaluating (2.18), we obtain the Hamiltonian (1.1). Notice that the Hamiltonian is Hermitian if ξ_{\pm} are imaginary and κ_{\pm} are real. The corresponding energy eigenvalues E are given by

$$E = \frac{\Lambda''(0)}{8(-1)^{N+1} \sinh \xi_- \sinh \xi_+}, \quad (2.19)$$

where $\Lambda(u)$ are eigenvalues of the transfer matrix.

The transfer matrix has the periodicity property

$$t(u + i\pi) = t(u), \quad (2.20)$$

as follows from (2.9), (2.14). Moreover, the transfer matrix has crossing symmetry

$$t(-u - \frac{i\pi}{2}) = t(u), \quad (2.21)$$

which can be proved using a generalization of the methods developed in the appendices of [26]. Finally, we note that the transfer matrix has the asymptotic behavior (for $\kappa_{\pm} \neq 0$)

$$t(u) \sim \kappa_- \kappa_+ i^N \frac{e^{u(4+2N)}}{2^{1+2N}} \mathbb{I} + \dots \quad \text{for} \quad u \rightarrow \infty. \quad (2.22)$$

²As is customary, we usually suppress the “quantum-space” subscripts $1, \dots, N$.

3 Inversion identity

Our main objective is to determine the eigenvalues $\Lambda(u)$ of the open-chain transfer matrix (2.15), from which the energy eigenvalues (2.19) immediately follow. We shall accomplish this using an exact inversion identity, which we first derive. A similar approach was used in [18, 19] for closed chains. This approach is based on the concept of fusion [20, 21].

The derivation of the inversion identity for the open XX spin chain/ 6-vertex free-Fermion model closely parallels the one for the 8-vertex free-Fermion model considered in [13]. For brevity, we shall often refer to these two models as the $\mathcal{N} = 2$ and $\mathcal{N} = 1$ models, respectively. The principal tool which we use to derive the inversion identity is the open-chain fusion formula obtained in [16]. We shall henceforth refer to this reference as I.

The matrix $R_{12}(u)$ at $u = -\rho = \frac{i\pi}{2}$ is proportional to the one-dimensional projector P_{12}^-

$$P_{12}^- = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (P_{12}^-)^2 = P_{12}^-. \quad (3.1)$$

As explained in I, from the corresponding degeneration of the (boundary) Yang-Baxter equation, one can derive identities which allow one to prove that *fused* (boundary) matrices satisfy *generalized* (boundary) Yang-Baxter equations.

The fused R matrix is given by (I 2.13)

$$R_{<12>3}(u) = P_{12}^+ R_{13}(u) R_{23}(u + \rho) P_{12}^+, \quad (3.2)$$

where $P_{12}^+ = \mathbb{I} - P_{12}^-$. An important observation is that the fused R matrix can be brought to the following upper triangular form by a similarity transformation

$$X_{12} R_{<12>3}(u) X_{12}^{-1} = \begin{pmatrix} s\mathbb{I} & * & * & * \\ 0 & t\sigma^z & -2t\sigma^z & 0 \\ 0 & 0 & -t\sigma^z & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.3)$$

where

$$s = \cosh^2 u, \quad t = i \cosh u \sinh u, \quad (3.4)$$

and the 4×4 matrix X is given by

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \quad (3.5)$$

It follows that the fused monodromy matrices (I 4.7), (I 5.4), (I 5.5)

$$\begin{aligned} T_{<12>}(u) &= R_{<12>N}(u) \cdots R_{<12>1}(u), \\ \hat{T}_{<12>}(u + \rho) &= R_{<12>1}(u) \cdots R_{<12>N}(u), \end{aligned} \quad (3.6)$$

also become triangular by the same transformation,

$$\begin{aligned} X_{12} T_{<12>}(u) X_{12}^{-1} &= \begin{pmatrix} s^N \mathbb{I} & * & * & * \\ 0 & t^N F & ((-1)^N - 1)t^N F & 0 \\ 0 & 0 & (-t)^N F & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= X_{12} T_{<12>}(u + \rho) X_{12}^{-1}, \end{aligned} \quad (3.7)$$

where $F = \prod_{i=1}^N \sigma_i^z$.

The corresponding fused K matrices are given by (I 3.5), (I 3.9)

$$\begin{aligned} K_{<12>}^-(u) &= P_{12}^+ K_1^-(u) R_{12}(2u + \rho) K_2^-(u + \rho) P_{12}^+, \\ K_{<12>}^+(u) &= \{P_{12}^+ K_1^+(u)^{t_1} R_{12}(-2u - 3\rho) K_2^+(u + \rho)^{t_2} P_{12}^+\}^{t_{12}}, \end{aligned} \quad (3.8)$$

since $M = V^t V = \mathbb{I}$.

Unlike the $\mathcal{N} = 1$ case [13], the similarity transformation does *not* bring also the fused K matrices to upper triangular form.³ Nevertheless, the transformed fused K matrices are “almost” triangular

$$X_{12} K_{<12>}^\mp(u) X_{12}^{-1} = \begin{pmatrix} m_1^\mp & * & * & * \\ 0 & m_2^\mp & m_5^\mp & 0 \\ 0 & m_4^\mp & m_3^\mp & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.9)$$

where

$$\begin{aligned} m_1^\mp &= \pm \frac{1}{2} \sinh 2u [\cosh 2u - \cosh 2\xi_\mp + 2\kappa_\mp^2 \sinh^2 2u], \\ m_2^\mp &= -\frac{i}{2} \sinh 2u [\sinh 2(u \pm \xi_\mp) - 8\kappa_\mp^2 \cosh u \sinh^3 u], \\ m_3^\mp &= -\frac{i}{2} \sinh 2u [\sinh 2(u \mp \xi_\mp) + 8\kappa_\mp^2 \cosh^3 u \sinh u], \\ m_4^\mp &= i\kappa_\mp^2 \cosh 2u \sinh^2 2u, \\ m_5^\mp &= \pm \frac{i}{2} \sinh 4u \sinh 2\xi_\mp. \end{aligned} \quad (3.10)$$

³It is not possible to simultaneously triangularize both $R_{<12>3}(u)$ and $K_{<12>}^-(u)$, since their commutator is not nilpotent; i.e., $[R_{<12>3}(u), K_{<12>}^-(u)]^n \neq 0$. A monograph on the general problem of simultaneous triangularization has recently been published [27].

The fused transfer matrix $\tilde{t}(u)$ is given by (I 4.5), (I 4.6)

$$\tilde{t}(u) = \text{tr}_{12} K_{<12>}^+(u) T_{<12>}(u) K_{<12>}^-(u) \hat{T}_{<12>}(u + \rho). \quad (3.11)$$

With the help of the results (3.7), (3.9), (3.10), we now obtain the remarkable result that the fused transfer matrix is proportional to the identity matrix,

$$\tilde{t}(u) = \tilde{\Lambda}(u) \mathbb{I}, \quad (3.12)$$

where

$$\begin{aligned} \tilde{\Lambda}(u) &= s^{2N} m_1^+ m_1^- + t^{2N} \left[m_2^+ m_2^- + m_3^+ m_3^- + (-1)^N (m_4^+ m_5^- + m_4^- m_5^+) \right. \\ &\quad \left. + (1 - (-1)^N) (m_3^+ m_4^- + m_3^- m_4^+ - m_2^+ m_4^- - m_2^- m_4^+ + 2m_4^- m_4^+) \right]. \end{aligned} \quad (3.13)$$

A similar result also holds for the $\mathcal{N} = 1$ case [13].

The fusion formula is given by (I 4.17), (I 5.1)

$$t(u) t(u + \rho) = \frac{1}{\zeta(2u + 2\rho)} \left[\tilde{t}(u) + \Delta \{K^+(u)\} \Delta \{K^-(u)\} \delta \{T(u)\} \delta \{\hat{T}(u)\} \right], \quad (3.14)$$

where the transfer matrix $t(u)$ is given by (2.15) (see also (I 4.1), (I 4.2)), and the quantum determinants [28, 21] are given by (I 4.15), (I 5.3), (I 5.7)

$$\begin{aligned} \delta \{T(u)\} &= \delta \{\hat{T}(u)\} = \zeta(u + \rho)^N, \\ \Delta \{K^-(u)\} &= \text{tr}_{12} \{P_{12}^- K_1^-(u) R_{12}(2u + \rho) K_2^-(u + \rho) V_1 V_2\}, \\ \Delta \{K^+(u)\} &= \text{tr}_{12} \{P_{12}^- V_1 V_2 K_2^+(u + \rho) R_{12}(-2u - 3\rho) K_1^+(u)\}. \end{aligned} \quad (3.15)$$

It follows from (3.12)-(3.15) that the transfer matrix obeys an exact inversion identity

$$t(u) t(u - \frac{i\pi}{2}) = f(u) \mathbb{I}. \quad (3.16)$$

where the function $f(u)$ is given by

$$f(u) = \tanh^2 2u \left[g_1(u) \cosh^{4N} u + g_2(u) \sinh^{4N} u + g_3(u) \sinh^{2N} u \cosh^{2N} u \right], \quad (3.17)$$

with

$$\begin{aligned} g_1(u) &= \frac{1}{4} (\cosh 2u - \cosh 2\xi_- + 2\kappa_-^2 \sinh^2 2u) (\cosh 2u - \cosh 2\xi_+ + 2\kappa_+^2 \sinh^2 2u) \\ &= 16\kappa_+^2 \kappa_-^2 \cosh(u + i\eta_-) \cosh(u - i\eta_-) \cosh(u + i\eta_+) \cosh(u - i\eta_+) \\ &\quad \times \cosh(u + \vartheta_-) \cosh(u - \vartheta_-) \cosh(u + \vartheta_+) \cosh(u - \vartheta_+), \end{aligned}$$

$$\begin{aligned}
g_2(u) &= g_1\left(u + \frac{i\pi}{2}\right), \\
g_3(u) &= 2\kappa_+^2 \kappa_-^2 \left[1 + (-1)^N + \sinh^2 2u\right] \sinh^2 2u + (-1)^N \left[(\kappa_+^2 \cosh 2\xi_- + \kappa_-^2 \cosh 2\xi_+) \sinh^2 2u\right. \\
&\quad \left. + \frac{1}{2}(\cosh 2\xi_- \cosh 2\xi_+ \sinh^2 2u - \sinh 2\xi_- \sinh 2\xi_+ \cosh^2 2u)\right] \\
&= 2\kappa_+^2 \kappa_-^2 \left[\sinh^2 2u \cosh^2 2u + (-1)^N \left(\sin 2\eta_- \sinh 2\vartheta_- \sin 2\eta_+ \sinh 2\vartheta_+ \cosh^2 2u\right.\right. \\
&\quad \left.\left. + \cos 2\eta_- \cosh 2\vartheta_- \cos 2\eta_+ \cosh 2\vartheta_+ \sinh^2 2u\right)\right]. \tag{3.18}
\end{aligned}$$

The inversion identity (3.16)-(3.18) is the first main result of our paper. We have checked it numerically up to $N = 3$.

4 Eigenvalues and Bethe Ansatz Equations

Having obtained the inversion identity, we now use it to determine the eigenvalues of the transfer matrix. The commutativity relation (2.17) implies that the transfer matrix has eigenstates $|\Lambda\rangle$ which are independent of u ,

$$t(u)|\Lambda\rangle = \Lambda(u)|\Lambda\rangle, \tag{4.1}$$

where $\Lambda(u)$ are the corresponding eigenvalues. Acting on $|\Lambda\rangle$ with the inversion identity, we obtain the corresponding identity for the eigenvalues

$$\Lambda(u) \Lambda\left(u - \frac{i\pi}{2}\right) = f(u). \tag{4.2}$$

Similarly, it follows from (2.20) and (2.21) that the eigenvalues have the periodicity and crossing properties

$$\Lambda(u + i\pi) = \Lambda(u), \quad \Lambda\left(-u - \frac{i\pi}{2}\right) = \Lambda(u). \tag{4.3}$$

Finally, (2.22) implies the asymptotic behavior

$$\Lambda(u) \sim \kappa_- \kappa_+ i^N \frac{e^{u(4+2N)}}{2^{1+2N}} + \dots \quad \text{for} \quad u \rightarrow \infty. \tag{4.4}$$

We shall assume that the eigenvalues have the form

$$\Lambda(u) = \rho \sinh 2u \prod_{j=0}^N \sinh(u - u_j) \cosh(u + u_j), \tag{4.5}$$

where u_j and ρ are (u -independent) parameters which are to be determined. Indeed, this expression satisfies the periodicity and crossing properties (4.3), and it has the correct asymptotic behavior (4.4) provided that we set

$$\rho = i^N 4\kappa_- \kappa_+. \quad (4.6)$$

We now substitute the Ansatz (4.5) into the inversion identity (4.2), and obtain

$$(-1)^N \rho^2 \sinh^2 2u \prod_{j=0}^N \frac{1}{4} \sinh 2(u - u_j) \sinh 2(u + u_j) = f(u). \quad (4.7)$$

Recalling the explicit expression (3.17),(3.18) for $f(u)$, we verify that both sides of the equation have the same asymptotic behavior $\sim e^{u(8+4N)}$ for $u \rightarrow \infty$. Since the LHS has zeros $\pm u_j$, these must be zeros of $f(u)$. That is, ⁴

$$g_1(u_j) \cosh^{4N} u_j + g_2(u_j) \sinh^{4N} u_j + g_3(u_j) \sinh^{2N} u_j \cosh^{2N} u_j = 0. \quad (4.8)$$

Dividing by $\cosh^{4N} u_j$, we obtain

$$g_2(u_j) \tanh^{4N} u_j + g_3(u_j) \tanh^{2N} u_j + g_1(u_j) = 0. \quad (4.9)$$

Regarding (4.9) as the quadratic equation

$$g_2 x^2 + g_3 x + g_1 = 0 \quad (4.10)$$

in the variable $x = \tanh^{2N} u_j$, we conclude that the parameters u_j satisfy the Bethe Ansatz equations

$$\tanh^{2N} u_j = \frac{h(u_j)}{g_2(u_j)}, \quad (4.11)$$

where the function $h(u)$ is defined by

$$h(u) = \frac{-g_3(u) \pm \sqrt{g_3(u)^2 - 4g_1(u) g_2(u)}}{2}. \quad (4.12)$$

The square root in (4.12) can be eliminated by making an appropriate change of variables. Indeed, with the help of (3.18), one can show that

$$\begin{aligned} h(u) = & \kappa_+^2 \kappa_-^2 \left\{ -\sinh^2 2u \cosh^2 2u - (-1)^N (\gamma_1 \cosh^2 2u + \gamma_2 \sinh^2 2u) \right. \\ & \left. \pm \sinh 2u \cosh 2u \sqrt{\alpha \sinh^2 2u + \beta} \right\}, \end{aligned} \quad (4.13)$$

⁴The three functions $g_i(u)$, $i = 1, 2, 3$, are even functions of u . Hence, if u_j is a root of $f(u)$, then so is $-u_j$.

where

$$\begin{aligned}
\alpha &= \frac{1}{2} (\cos 4\eta_- + \cos 4\eta_+ + \cosh 4\vartheta_- + \cosh 4\vartheta_+) + (-1)^N [\cos 2(\eta_- + \eta_+) \cosh 2(\vartheta_- - \vartheta_+) \\
&\quad + \cos 2(\eta_- - \eta_+) \cosh 2(\vartheta_- + \vartheta_+)], \\
\beta &= [\sin 2\eta_- \sinh 2\vartheta_- + (-1)^N \sin 2\eta_+ \sinh 2\vartheta_+]^2 - \frac{1}{4} [\sin 2(\eta_- + \eta_+) \sinh 2(\vartheta_- - \vartheta_+) \\
&\quad + \sin 2(\eta_- - \eta_+) \sinh 2(\vartheta_- + \vartheta_+)]^2, \\
\gamma_1 &= \sin 2\eta_- \sinh 2\vartheta_- \sin 2\eta_+ \sinh 2\vartheta_+, \\
\gamma_2 &= \cos 2\eta_- \cosh 2\vartheta_- \cos 2\eta_+ \cosh 2\vartheta_+. \tag{4.14}
\end{aligned}$$

Let us change from the spectral parameter u to the new spectral parameter v defined by

$$\sinh 2u = i \operatorname{sn} 2v, \quad \cosh 2u = \operatorname{cn} 2v, \tag{4.15}$$

where the modulus k of the Jacobian elliptic functions is given by

$$k^2 = \frac{\alpha}{\beta}. \tag{4.16}$$

With the help of the identities (see, e.g., [29])

$$\operatorname{cn}^2 z + \operatorname{sn}^2 z = 1, \quad \operatorname{dn}^2 z + k^2 \operatorname{sn}^2 z = 1, \tag{4.17}$$

one can see that the function $h(u)$ can be reexpressed as

$$h(u) = \kappa_+^2 \kappa_-^2 \left\{ \operatorname{sn}^2 2v \operatorname{cn}^2 2v - (-1)^N (\gamma_1 \operatorname{cn}^2 2v - \gamma_2 \operatorname{sn}^2 2v) \pm i \operatorname{sn} 2v \operatorname{cn} 2v \operatorname{dn} 2v \sqrt{\beta} \right\} \tag{4.18}$$

Hence, the Bethe Ansatz solution (4.5), (4.6), (4.11) can be reformulated as

$$\Lambda(u) = \left(-\frac{1}{2}\right)^{N-1} \kappa_+ \kappa_- \operatorname{sn} 2v \prod_{j=0}^N (\operatorname{sn} 2v - \operatorname{sn} 2v_j), \tag{4.19}$$

where the parameters v_j satisfy

$$\left(\frac{\operatorname{cn} 2v_j - 1}{\operatorname{cn} 2v_j + 1} \right)^N = \frac{\operatorname{sn}^2 2v_j \operatorname{cn}^2 2v_j - (-1)^N (\gamma_1 \operatorname{cn}^2 2v_j - \gamma_2 \operatorname{sn}^2 2v_j) \pm i \operatorname{sn} 2v_j \operatorname{cn} 2v_j \operatorname{dn} 2v_j \sqrt{\beta}}{(\operatorname{cn} 2v_j - \operatorname{cn} 2i\eta_-)(\operatorname{cn} 2v_j - \operatorname{cn} 2\vartheta_-)(\operatorname{cn} 2v_j - \operatorname{cn} 2i\eta_+)(\operatorname{cn} 2v_j - \operatorname{cn} 2\vartheta_+)}. \tag{4.20}$$

This Bethe Ansatz solution is the second main result of our paper. This result passes several tests. Indeed, for $N = 0, 1$, the eigenvalues agree with those obtained by direct diagonalization of the transfer matrix. Moreover, as discussed in the following section, our solution is similar to the known one [10, 1] for the case of diagonal boundary terms.

5 Special cases

For generic values of the boundary parameters, the Bethe Ansatz solution presented in the previous section is formulated in terms of Jacobian elliptic functions. However, for modulus $k = 0$ or $k = 1$, these elliptic functions degenerate into ordinary trigonometric or hyperbolic functions. Equivalently, the argument of the square root in (4.13) then becomes a perfect square, and so the square root effectively disappears. We now briefly consider some of these special cases.

5.1 Diagonal case

In the limit $\kappa_{\pm} \rightarrow 0$, the K matrices (2.11), (2.12) become diagonal, and therefore so do the boundary terms in the Hamiltonian (1.1). The transfer matrix $t(u)$ now commutes with the operator $F = \prod_{i=1}^N \sigma_i^z$, and hence, both operators can be simultaneously diagonalized. Denoting the corresponding eigenstates by $|\Lambda^{(\pm)}\rangle$, we have

$$\begin{aligned} t(u)|\Lambda^{(\pm)}\rangle &= \Lambda^{(\pm)}(u)|\Lambda^{(\pm)}\rangle, \\ F|\Lambda^{(\pm)}\rangle &= \pm|\Lambda^{(\pm)}\rangle. \end{aligned} \quad (5.1)$$

The transfer matrix now has the asymptotic behavior

$$t(u) \sim \rho \frac{e^{u(2+2N)}}{2^{1+2N}} F + \dots \quad \text{for} \quad u \rightarrow \infty, \quad (5.2)$$

where

$$\rho = \begin{cases} i \cosh(\xi_+ - \xi_-) & \text{for } N = \text{even} \\ i \sinh(\xi_+ - \xi_-) & \text{for } N = \text{odd} \end{cases}. \quad (5.3)$$

The eigenvalues are given by

$$\Lambda^{(\pm)}(u) = \pm \rho \sinh 2u \prod_{j=1}^N \sinh(u - u_j) \cosh(u + u_j), \quad (5.4)$$

which is similar to (4.5) except with one less root.

In terms of the boundary parameters $\eta_{\pm}, \vartheta_{\pm}$ (2.13), the limit $\kappa_{\pm} \rightarrow 0$ corresponds to

$$\eta_{\pm} = i\xi_{\pm} - \frac{\pi}{2}, \quad e^{-\vartheta_{\pm}} = \frac{1}{\kappa_{\pm}} \rightarrow \infty. \quad (5.5)$$

In this limit the function $h(u)$ becomes equal (for $N = \text{even}$) to

$$h(u) = -\sinh(u \mp \xi_-) \cosh(u \mp \xi_-) \sinh(u \pm \xi_+) \cosh(u \pm \xi_+). \quad (5.6)$$

Moreover, the Bethe Ansatz equations (4.11) become

$$\tanh^{2N} u_j = -\frac{\sinh(u_j \mp \xi_-) \sinh(u_j \pm \xi_+)}{\cosh(u_j \pm \xi_-) \cosh(u_j \mp \xi_+)} . \quad (5.7)$$

These results are similar to those obtained previously [10, 1] for the XXZ chain. The two choices of signs correspond to the two possible pseudovacua – either all spins up or all spins down.

5.2 Nondiagonal cases

Within the space of boundary parameters, there are various submanifolds, such as

$$\eta_- - \eta_+ \pm i(\vartheta_- - \vartheta_+) = \frac{\pi}{2} \frac{1 + (-1)^N}{2} , \quad (5.8)$$

for which $\alpha = k = 0$.

As a simple example, let us consider the particular case $N = \text{odd}$, $\eta_- = \eta_+ \equiv \eta$, $\vartheta_- = \vartheta_+ \equiv \vartheta$, for which also $\beta = 0$. The function $h(u)$ becomes equal to

$$h(u) = -\kappa_+^2 \kappa_-^2 \sinh 2(u - i\eta) \sinh 2(u + i\eta) \sinh 2(u - \vartheta) \sinh 2(u + \vartheta) . \quad (5.9)$$

We then obtain the Bethe Ansatz equations

$$\tanh^{2N} u_j = -\coth(u_j + i\eta) \coth(u_j - i\eta) \coth(u_j + \vartheta) \coth(u_j - \vartheta) . \quad (5.10)$$

6 Discussion

This work raises a number of interesting questions, some of which we list below:

We have seen that the doubly-periodic functions in the Bethe Ansatz solution degenerate into singly-periodic functions for special values of the boundary parameters. In [5], important simplifications are also found to occur for special values of the boundary parameters. It is likely that these two observations are related.

As we have emphasized, an exact inversion identity for the XX (or $\mathcal{N} = 2$) case is made possible by the key fact that the fused transfer matrix is proportional to the identity matrix. A similar result also holds for the $\mathcal{N} = 1$ case [13]. It would be interesting to better understand the relation of this phenomenon to the free-Fermion condition.

The model (1.1) is not the most general integrable open XX chain. Indeed, the most general solution [2, 3] of the XXZ boundary Yang-Baxter equation has the off-diagonal terms

$\kappa_{\pm}^{(1)} \sinh 2u$ and $\kappa_{\pm}^{(2)} \sinh 2u$, while here we have restricted to the special case $\kappa_{\pm}^{(1)} = \kappa_{\pm}^{(2)} \equiv \kappa_{\pm}$. (See Eqs. (2.11), (2.12).) However, we do not expect that the more general case will lead to new significant complications. In particular, we expect that the same approach can be used to derive an exact inversion identity and to obtain the corresponding Bethe Ansatz solution.

Since the model (1.1) has various boundary parameters, its phase diagram is likely to have a rich structure. Our exact Bethe Ansatz solution should provide a means of exploring these phases. Moreover, as mentioned in the Introduction, this solution opens the way to formulating the thermodynamic Bethe Ansatz equations for integrable $\mathcal{N} = 2$ supersymmetric quantum field theories with boundary [14, 15], as was done for the case of $\mathcal{N} = 1$ supersymmetry in [13]. Finally, with the insight gained from the XX chain, it might now be possible to finally solve the open XXZ chain with nondiagonal boundary terms.

We hope to report on some of these problems in future publications.

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